

ON AN EXTENSION OF THE BLASCHKE-SANTALÓ INEQUALITY

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ABSTRACT. Let K be a convex body and K° its polar body. Call $\phi(K) = \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy$. It is conjectured that $\phi(K)$ is maximum when K is the euclidean ball. In particular this statement implies the Blaschke-Santaló inequality. We verify this conjecture when K is restricted to be a p -ball.

1. INTRODUCTION AND NOTATION

A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. For every convex body, its polar set is defined

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^n . Note that if $0 \in \text{int}K$ then K° is a convex body.

For $p \in [1, \infty]$, let us denote by B_p^n the unit ball of the p -norm. It is:

$$B_p^n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1 \right\} \quad B_\infty^n = \{x \in \mathbb{R}^n : \max |x_i| \leq 1\}.$$

It is well known that the polar body of B_p^n is B_q^n where q is the dual exponent of p ($\frac{1}{p} + \frac{1}{q} = 1$). Along this paper q will always denote the dual exponent of p .

Given two symmetric convex bodies $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, for any $p \in [1, \infty]$ they define a symmetric convex body $A \times_p B \subset \mathbb{R}^{n+m}$ which is the unit ball of the norm given by

$$\|(x_1, x_2)\|_{A \times_p B}^p = \|x_1\|_A^p + \|x_2\|_B^p \quad \|(x_1, x_2)\|_{A \times_\infty B} = \max\{\|x_1\|_A, \|x_2\|_B\}.$$

Note that the polar body of $A \times_p B$ is $A^\circ \times_q B^\circ$ and $B_p^n = B_p^{n-1} \times_p [-1, 1]$.

A convex body K is said to be in isotropic position if it has volume 1 and satisfies the following two conditions:

- $\int_K x dx = 0$ (center of mass at 0)
- $\int_K \langle x, \theta \rangle^2 dx = L_K^2 \quad \forall \theta \in S^{n-1}$

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where L_K is a constant independent of θ , which is called the isotropy constant of K .

We will use the notation \tilde{K} for $|K|^{-\frac{1}{n}}K$.

Given a centrally symmetric convex body K , we call

$$\phi(K) = \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy.$$

Note that $\phi(K) = \phi(TK)$ for all $T \in GL(n)$. It is conjectured in [5] that $\phi(K)$ is maximized by ellipsoids. It is, for every symmetric convex body $K \subset \mathbb{R}^n$

$$(1) \quad \phi(K) \leq \phi(B_2^n) = \frac{n}{(n+2)^2}.$$

Remark. We can also define the functional ϕ when K is not symmetric. When K is a regular simplex with its center of mass at the origin, it is easy to compute that $\phi(K) = \phi(B_2^n)$.

The Blaschke-Santaló inequality [6] says that for every symmetric convex body K

$$|K||K^\circ| \leq |B_2^n|^2.$$

The conjecture (1) is stronger than the Blaschke-Santaló inequality since

$$\frac{n|K|^{\frac{2}{n}}|K^\circ|^{\frac{2}{n}}}{(n+2)^2|B_2^n|^{\frac{4}{n}}} \leq \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy.$$

This fact is a consequence of Lemma 6 in [2]. In [3], Ball proved that for 1-unconditional bodies

$$\int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy \leq \frac{n|B_2^n|^2}{(n+2)^2}$$

and suggested that this inequality might be true for every convex body. This assertion is slightly weaker than the conjecture in [5], which is not known to be true even for 1-unconditional bodies. In section 2 we are going to prove that the conjecture is true if we restrict K to be a p -ball, for some $p \geq 1$. We state this as a theorem:

Theorem 1.1. *Among the p -balls, the functional ϕ is maximized for the euclidean ball.*

$$\max_{p \in [1, \infty]} \phi(B_p^n) = \phi(B_2^n) = \frac{n}{(n+2)^2}.$$

The conjecture (1) is also stronger than the hyperplane conjecture, which says that there exists an absolute constant C such that for every isotropic convex body $L_K < C$. It can be proved that $\phi(K)$ is bounded from below by $\frac{c_1}{n}$, where c_1 is an absolute constant. If there exists an absolute constant c_2 such that $\phi(K) \leq \frac{c_2}{n}$, then the hyperplane conjecture would be true, since

$$nL_K^2 L_{K^\circ}^2 \leq \frac{\phi(K)}{|K|^{\frac{2}{n}}|K^\circ|^{\frac{2}{n}}} \leq cn^2 \phi(K)$$

where c is an absolute constant.

In case that \widetilde{K} and \widetilde{K}° are both isotropic then $\phi(K) = n|K|^{\frac{2}{n}}|K^\circ|^{\frac{2}{n}}L_K^2L_{K^\circ}^2$ and the conjecture $\phi(K) \sim \frac{1}{n}$ is equivalent to the hyperplane conjecture. This is the case of 1-symmetric bodies, for which the hyperplane conjecture is known to be true (A convex body is 1-symmetric if it is invariant under reflections in the coordinate hyperplanes and under permutations of the coordinates).

We will say that a symmetric convex body $K \subset \mathbb{R}^n$ is a revolution body if there exists $\theta \in S^{n-1}$ and a concave function $r(t)$ such that for every $t \in [-h_K(\theta), h_K(\theta)]$ $K \cap (t\theta + \theta^\perp) = r(t)B_2^{n-1}$, where $h_K(\theta)$ is the support function of K :

$$h_K(\theta) = \max\{\langle x, \theta \rangle : x \in K\}.$$

In section 3 we will prove that there exists an absolute constant C such that whenever K is a symmetric convex body of revolution, $\phi(K) \leq \frac{C}{n}$.

Along this paper, ψ will always denote the logarithmic derivative of the Gamma function. We will make use of the following identity on the derivatives of ψ , known as polygamma functions:

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty t^n \frac{e^{-xt}}{1 - e^{-t}} dt.$$

The letters C, c_1, c_2, \dots will always denote absolute constants which do not depend on the dimension.

2. THE p -BALLS

In this section we are going to prove theorem 1.1. We will obtain it as a consequence of the following

Theorem 2.1. *For every $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, $p \in [1, \infty]$*

$$\phi(A \times_p B) = f(n, n+m, p)\phi(A) + f(m, n+m, p)\phi(B)$$

where

$$f(y_1, y_2, p) = \begin{cases} \frac{(y_1+2)^2 y_2^2 \Gamma(\frac{y_1+2}{p}) \Gamma(\frac{y_1+2}{q}) \Gamma(\frac{y_2}{p}) \Gamma(\frac{y_2}{q})}{y_1^2 (y_2+2)^2 \Gamma(\frac{y_2+2}{p}) \Gamma(\frac{y_2+2}{q}) \Gamma(\frac{y_1}{p}) \Gamma(\frac{y_1}{q})} & p \neq 1, \infty \\ \frac{(y_1+2) y_2 \Gamma(y_1+2) \Gamma(y_2)}{y_1 (y_2+2) \Gamma(y_2+2) \Gamma(y_1)} & p = 1, \infty \end{cases}$$

attains its maximum when $p = 2$, for every $0 < y_1 < y_2$.

Proof. First of all we are going to prove that for every fixed $0 < y_1 < y_2$, the function defined on $[0, 1]$ like $f_1(x) = f(y_1, y_2, \frac{1}{x})$ attains its maximum in $x = \frac{1}{2}$. It is easy to check that $f_1(0) = f_1(1) < f_1(\frac{1}{2})$. f_1 has got a maximum in $x = \frac{1}{2}$ if and only if $\log f_1$ has got a maximum in $x = \frac{1}{2}$.

Since $f_1(x) = f_1(1-x)$, it is enough to prove that $\log f_1$ is increasing in $(0, \frac{1}{2})$. Now, if we call

$$F(x, y) = (y+2)[\psi((y+2)x) - \psi((y+2)(1-x))] - y[\psi(yx) - \psi(y(1-x))]$$

we have that

$$\begin{aligned} (\log f_1)'(x) &= (y_1 + 2)[\psi((y_1 + 2)x) - \psi((y_1 + 2)(1 - x))] - y_1[\psi(y_1 x) - \psi(y_1(1 - x))] \\ &\quad - (y_2 + 2)[\psi((y_2 + 2)x) - \psi((y_2 + 2)(1 - x))] + y_2[\psi(y_2 x) - \psi(y_2(1 - x))] \\ &= F(x, y_1) - F(x, y_2). \end{aligned}$$

So it is enough to prove that for every fixed $x \in (0, \frac{1}{2})$, $F(x, y)$ is decreasing in $y \in (0, \infty)$. Hence we compute

$$\begin{aligned} \frac{\partial F}{\partial y}(x, y) &= \psi((y + 2)x) - \psi((y + 2)(1 - x)) - \psi(yx) + \psi(y(1 - x)) \\ &\quad + (y + 2)x\psi'((y + 2)x) - (y + 2)(1 - x)\psi'((y + 2)(1 - x)) \\ &\quad - yx\psi'(yx) + y(1 - x)\psi'(y(1 - x)). \end{aligned}$$

We call this last quantity $G(x, y)$ and we will see that $G(x, y) < 0$ if $x \in (0, \frac{1}{2})$ and $G(x, y) > 0$ if $x \in (\frac{1}{2}, 1)$. Notice that $G(\frac{1}{2}, y) = 0$, so we just need to check that for every fixed $y > 0$, $G(x, y)$ is increasing in x . Computing its derivative we obtain

$$\begin{aligned} \frac{\partial G}{\partial x}(x, y) &= 2(y + 2)[\psi'((y + 2)x) + \psi'((y + 2)(1 - x))] \\ &\quad + (y + 2)^2[x\psi''((y + 2)x) + (1 - x)\psi''((y + 2)(1 - x))] \\ &\quad - 2y[\psi'(yx) + \psi'(y(1 - x))] - y^2[x\psi''(yx) + (1 - x)\psi''(y(1 - x))] \\ &= H(x, y + 2) - H(x, y). \end{aligned}$$

where we have called $H(x, y)$ the following function

$$H(x, y) = 2y[\psi'(yx) + \psi'(y(1 - x))] + y^2[x\psi''(yx) + (1 - x)\psi''(y(1 - x))].$$

Hence, if for every fixed $x \in (0, 1)$ $H(x, y)$ is increasing in y , then so it is $G(x, y)$ in x for fixed y and the theorem is proved. In order to prove this, we need the following result concerning the ψ function whose proof can be found in [1]. We will write it here for the sake of completeness:

Proposition 2.1. *The function $f(x) = x^2\psi'(x)$ is convex in the interval $(0, \infty)$.*

Proof. The second derivative of f is

$$f''(x) = 2\psi'(x) + 4x\psi''(x) + x^2\psi'''(x).$$

Using the integral representation of the derivatives of ψ this is equal to

$$\begin{aligned} f''(x) &= \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}} (2t - 4xt^2 + x^2t^3) dt \\ &= \int_0^\infty \frac{t}{1 - e^{-t}} \frac{d^2}{dt^2} (t^2 e^{-xt}) dt \\ &= \int_0^\infty \frac{d^2}{dt^2} \left(\frac{t}{1 - e^{-t}} \right) t^2 e^{-xt} dt \end{aligned}$$

which is positive since the function $\frac{t}{1 - e^{-t}}$ is convex in the interval $(0, \infty)$. \square

Now, for every $x \in (0, 1)$, $y > 0$ we have that

$$\begin{aligned} \frac{\partial H}{\partial y}(x, y) &= 2\psi'(yx) + 4yx\psi''(yx) + y^2x^2\psi'''(yx) \\ &\quad + 2\psi'(y(1-x)) + 4y(1-x)\psi''(y(1-x)) + y^2(1-x)^2\psi'''(y(1-x)) > 0 \end{aligned}$$

as a consequence of proposition 2.1 and this proves that $f(y_1, y_2, p) \leq f(y_1, y_2, 2)$ when $0 < y_1 < y_2$.

Let us prove now that

$$\phi(A \times_p B) = f(n, n+m, p)\phi(A) + f(m, n+m, p)\phi(B).$$

Assume that $p \neq 1, \infty$. We compute the volume of $A \times_p B$:

$$\begin{aligned} |A \times_p B| &= \int_A (1 - \|x_1\|_A^p)^{\frac{m}{p}} |B| dx_1 = \int_A \int_{\|x_1\|_A^p}^1 \frac{m}{p} (1-t)^{\frac{m}{p}-1} dt |B| dx_1 \\ &= \int_0^1 \int_{t^{\frac{1}{p}} A} \frac{m}{p} (1-t)^{\frac{m}{p}-1} |B| dx_1 dt = \frac{m}{p} |A| |B| \beta\left(\frac{m}{p} + 1, \frac{n}{p}\right) \\ &= \frac{nm}{p(n+m)} |A| |B| \beta\left(\frac{m}{p}, \frac{n}{p}\right). \end{aligned}$$

Since $(A \times_p B)^\circ = A^\circ \times_q B^\circ$, we have that

$$|(A \times_p B)^\circ| = \frac{nm}{q(n+m)} |A^\circ| |B^\circ| \beta\left(\frac{m}{q}, \frac{n}{q}\right).$$

From the symmetry of A and B we obtain that

$$\int_K \int_{K^\circ} \langle (x_1, x_2), (y_1, y_2) \rangle^2 dy dx = \int_K \int_{K^\circ} \langle x_1, y_1 \rangle^2 dy dx + \int_K \int_{K^\circ} \langle x_2, y_2 \rangle^2 dy dx$$

where we have called $K = A \times_p B$.

Let us compute these integrals:

$$\begin{aligned} &\int_K \int_{K^\circ} \langle x_1, y_1 \rangle^2 dy dx \\ &= \int_A \int_{A^\circ} \langle x_1, y_1 \rangle^2 (1 - \|x_1\|_A^p)^{\frac{m}{p}} (1 - \|y_1\|_{A^\circ}^q)^{\frac{m}{q}} |B| |B^\circ| dy_1 dx_1 \\ &= |B| |B^\circ| \int_A \int_{A^\circ} \langle x_1, y_1 \rangle^2 \int_{\|x_1\|_A^p}^1 \frac{m}{p} (1-t)^{\frac{m}{p}-1} dt \int_{\|y_1\|_{A^\circ}^q}^1 \frac{m}{q} (1-s)^{\frac{m}{q}-1} ds dy_1 dx_1 \\ &= |B| |B^\circ| \frac{m^2}{pq} \int_0^1 \int_0^1 \int_{t^{\frac{1}{p}} A} \int_{s^{\frac{1}{q}} A^\circ} \langle x_1, y_1 \rangle^2 (1-t)^{\frac{m}{p}-1} (1-s)^{\frac{m}{q}-1} dy_1 dx_1 ds dt \\ &= |B| |B^\circ| \frac{m^2}{pq} \beta\left(\frac{m}{p}, \frac{n+2}{p} + 1\right) \beta\left(\frac{m}{q}, \frac{n+2}{q} + 1\right) \int_A \int_{A^\circ} \langle x_1, y_1 \rangle^2 dy_1 dx_1 \\ &= |B| |B^\circ| \frac{m^2(n+2)^2}{pq(m+n+2)^2} \beta\left(\frac{m}{p}, \frac{n+2}{p}\right) \beta\left(\frac{m}{q}, \frac{n+2}{q}\right) \int_A \int_{A^\circ} \langle x_1, y_1 \rangle^2 dy_1 dx_1 \end{aligned}$$

and in the same way

$$\int_K \int_{K^\circ} \langle x_2, y_2 \rangle^2 dy dx =$$

$$= |A||A^\circ| \frac{n^2(m+2)^2}{pq(m+n+2)^2} \beta\left(\frac{n}{p}, \frac{m+2}{p}\right) \beta\left(\frac{n}{q}, \frac{m+2}{q}\right) \int_B \int_{B^\circ} \langle x_1, y_1 \rangle^2 dy_1 dx_1.$$

Now from the definition of ϕ and the identity $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we obtain the result. When $p = 1, \infty$ the theorem is proved in the same way. \square

3. REVOLUTION BODIES

In this section we are going to prove the following:

Theorem 3.1. *There exists an absolute constant C such that for every symmetric convex body of revolution K , $\phi(K) < \frac{C}{n}$.*

This is not a new result since A. Giannopoulos proved it in his PhD thesis but it was left unpublished. I would like to thank him for allowing me to add this result to this paper.

Proof. Since $\phi(TK) = \phi(K)$ for every $T \in GL(n)$, we can assume that

$$K = \{\bar{x} = (t, x) \in \mathbb{R}^n : t \in [-1, 1], |x| \leq r_1(t)\}$$

where $r_1(t)$ is a concave function such that $r_1(0) = 1$.

Then, K° is another revolution body

$$\begin{aligned} K^\circ &= \{\bar{y} = (s, y) \in \mathbb{R}^n : ts + r_1(t)|y| \leq 1, \forall t \in [-1, 1]\} \\ &= \{\bar{y} = (s, y) \in \mathbb{R}^n : s \in [-1, 1], |y| \leq r_2(s)\} \end{aligned}$$

where $r_2(s)$ is a concave function such that $r_2(0) = 1$

Let us now compute $\phi(K)$:

$$\begin{aligned} \phi(K) &= \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} (ts + \langle x, y \rangle)^2 d\bar{y} d\bar{x} \\ &= \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} t^2 s^2 + \langle x, y \rangle^2 d\bar{y} d\bar{x} \\ &= \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} t^2 s^2 d\bar{y} d\bar{x} \\ &\quad + \frac{1}{|K||K^\circ|} \int_{-1}^1 \int_{-1}^1 \int_{r_1(t)B_2^{n-1}} \int_{r_2(t)B_2^{n-1}} \langle x, y \rangle^2 dy dx ds dt \\ &= |K|^{\frac{2}{n}} |K^\circ|^{\frac{2}{n}} \int_{\tilde{K}} t^2 d\bar{x} \int_{\tilde{K}^\circ} s^2 d\bar{y} \\ &\quad + \frac{\int_{-1}^1 r_1(t)^{n+1} dt \int_{-1}^1 r_2(s)^{n+1} ds}{\int_{-1}^1 r_1(t)^{n-1} dt \int_{-1}^1 r_2(s)^{n-1} ds} \phi(B_2^{n-1}) \end{aligned}$$

Since $\max\{r_1(t), t \in [-1, 1]\} = r_1(0) = 1$ and $\max\{r_2(s), s \in [-1, 1]\} = r_2(0) = 1$, for every $t, s \in [-1, 1]$ we have that

- $r_1(t)^{n+1} \leq r_1(t)^{n-1}$
- $r_2(s)^{n+1} \leq r_2(s)^{n-1}$

and hence the second summand is bounded by $\phi(B_2^{n-1}) = \frac{n-1}{(n+1)^2}$.

To bound the first summand we will use the following well known result by Hensley[4]:

“There exist absolute constants c_1, c_2 such that for every symmetric convex body $K \subset \mathbb{R}^n$ with volume 1 and for every $\theta \in S^{n-1}$ ”

$$\frac{c_1}{|K \cap \theta^\perp|} \leq \left(\int_K \langle x, \theta \rangle^2 dx \right)^{\frac{1}{2}} \leq \frac{c_2}{|K \cap \theta^\perp|}.$$

Hence

$$\begin{aligned} \bullet \int_{\tilde{K}} t^2 d\bar{x} &\leq \frac{c|K|^2 \frac{n-1}{n}}{|K \cap e_1^\perp|^2} = \frac{c|K|^{2-\frac{2}{n}}}{|B_2^{n-1}|^2}, \\ \bullet \int_{\tilde{K}^\circ} s^2 d\bar{y} &\leq \frac{c|K^\circ|^2 \frac{n-1}{n}}{|K^\circ \cap e_1^\perp|^2} = \frac{c|K^\circ|^{2-\frac{2}{n}}}{|B_2^{n-1}|^2}. \end{aligned}$$

So, by Blaschke-Santaló inequality, the first summand is bounded by

$$\frac{c|K|^2|K^\circ|^2}{|B_2^{n-1}|^4} \leq \frac{c|B_2^n|^4}{|B_2^{n-1}|^4}.$$

Now, using the fact that $|B_2^n| = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ and Stirling's formula, we obtain that the first summand is bounded by $\frac{c}{n^2}$ and hence the theorem is proved. \square

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